N70-11539
NASA CR-106911

C A S C A D E   M O D E L   O F   T U R B U L E N C E

C. M. Tchen

The City College of The City University of New York

NGL-33-013-034

November 1969

CASCADE MODEL OF TURBULENCE

C. M. Tchen

The City College of The City University of New York

A B S T R A C T

A cascade concept is introduced. It decomposes a velocity fluctuation into a group of modes of large scales and a group of modes of smaller scales. The mutual interaction is responsible for the transfer of energy across the spectrum. The concept of cascade has the purpose of providing a mean of closure of the hierarchy of equations common to nonlinear systems. A new expression for the eddy viscosity is obtained, differing from that proposed by Heisenberg. It has the advantage of determining the Kolmogoroff law in the inertial subrange with an analytical numerical coefficient, and of calculating the spectral law in the viscous subrange. The latter law provides a cutoff in the spectrum and therefore secures the convergence of any high order velocity derivative. The cutoff follows the viscous subrange, and does not follow the inertial law, as predicted by some theories.

1. INTRODUCTION

In the equilibrium range, the development of the spectral distribution energy F(k) can be described by the following equation:

$$S(k) + 2\nu\int_0^k dk\, k^2 F(k) = \varepsilon$$

where ν is the molecular kinematic viscosity,

$$2\int_0^k dk\, k^2 F \equiv R^\circ$$

is the vorticity function, and νR_0 is the dissipation function. Further ε is the total energy dissipation

$$\varepsilon = \nu R^\circ(k=\infty)$$

Finally S(k) is the transfer function arising from the nonlinearity, and describing the flux of energy across the spectrum, i.e. the rate at which the portion of the spectrum with wave number less than k transfers energy to the remainder of the spectrum. By definition

$$S(0)=0, \qquad S(\infty)=0$$

this means that the nonlinear interactions transfer energy between Fourier components without dissipating energy. The determination of the structure of the transfer function is the crucial aim of any spectral theory of turbulence. There are the following phenomenological theories:

(i) Heisenberg's[1,2] theory takes

$$S(k) = \nu_k R^\circ$$

where

$$\nu_k = const\int_k^\infty dk\,(F/k^3)^{\frac{1}{2}}$$

[1]W. Heisenberg, Z. Physik 124, 628 (1948)

[2]W. Heisenberg, Proc. Roy. Soc. (London) 195, 503 (1948).

is an eddy viscosity.

(ii) Obukhov's[3] theory postulates

$$S = \text{const}\, R^{\circ\frac{1}{2}} \int_k^\infty dk\, F$$

(iii) Pao[4] and Kovasznay's[5] theory proposes

$$S = \text{const}\, \varepsilon \left(\varepsilon^{-2/3} k^{5/3} F\right)^n$$

Pao[4] uses n = 1, and Kovasznay[5] uses n = 3/2. Tennekes[6] notes that the case n = 3/4 gives the Heisenberg[1,2] law k^{-7} in the viscous subrange.

The theories (i) and (ii) are based on the idea that the small eddies act like an eddy viscosity on the big eddies. The theory (iii) has no simple physical analogue.

[3]A. M. Obukhov, C.R. Acad. Sci. USSR 32, 19 (1941); Izv. Akad. Nauk, USSR, Ser. Geogr. i. Geofiz. 5, 453 (1941), (translation issued by Ministry of Supply, United Kingdom, as P21109T).

[4]Y. H. Pao, Phys. Fluids 8, 1063 (1965).

[5]L. S. G. Kovasznay, J. Aeron Sci. 15, 745 (1948).

[6]H. Tennekes, Phys. Fluids 11, 246 (1968).

All the 3 theories agree with the Kolmogoroff[7] law of $k^{-5/3}$ in the inertial subrange, as required by dimensional conditions. The Kolmogoroff[7] law is also confirmed by Onsager[8,9] and von Weizsäcker[10].

As mentioned above, the solution k^{-7} of Heisenberg[1,2] bears the difficulty of a divergence in the mean square velocity derivatives of arbitrarily high orders. Pao's[4] solution finds an exponential decay as a tail following the $k^{-5/3}$ inertial law. The exponential cutoff is a superior choice, but one rather expects that a cutoff should be attached to the viscous law. The exponential tail predicted by Pao[4] varies as exp $(-k^{4/3})$ rather than exp $(-k^2)$, as suggested by Batchelor[11], Saffman[12] and Novikov[13]. In supplementing the above phenomenological theories, several physical models[13,14,15] have been advanced attempting the formulation of a dynamics of the interactions. The theories are confronted with several difficult questions:

(a) To devise an approximation of closure of the hierarchy of equations inherent in any nonlinear system.

(b) To formulate the dynamics of interactions between the modes, and to derive the transfer function.

(c) To derive the structure of the eddy viscosity, as a transport property representing the statistical effect of fluctuations of small scales upon larger ones.

(d) To derive the inertial and viscous laws as solutions of the equation of energy spectrum.

(e) To show the existence of an exponential tail, arising from the viscous cutoff of the spectrum at infinitely large wave numbers.

If those fundamental questions, which enter in the physical models, cannot find their answers from the phenomenological theories (i) - (iii), the recent analytical theories of turbulence can also not lend much help.

---

[7]A. N. Kolmogoroff, C. R. Acad. Sci. USSR 30, 301 (1941).

[8]L. Onsager, Phys. Rev. 68, 286 (1945).

[9]L. Onsager, Nuovo Cimento Suppl. 6, 279 (1949).

[10]C. F. von Weizsäcker, Z. Physik 124, 614 (1948).

[11]G. K. Batchelor, J. Fluid Mech. 9, 113 (1959).

[12]P. G. Saffman, J. Fluid Mech. 16, 546 (1963).

[13]E. A. Novikov, Dokl. Akad. Nauk USSR 139, 331 (1961) [Sov. Phys. - Doklady 6, 571 (1961)].

[13]C. E. Leith, Phys. Fluid 10, 1409 (1967).

[14]C. E. Leith, Phys. Fluid 11, 1612 (1968).

[15]E. N. Parker, Phys. Fluid 12, 1592 (1969).

In a recent survey of such theories, Orszag[16] commented that they are based on mostly unsatisfactory closure approximations, and therefore often fail to predict even a proper Kolmogoroff[7] inertial range spectrum. The extension of these theories to more complicate types of turbulence, e.g. plasma turbulence and magnetohydrodynamic turbulence will involve even greater difficulties. On the other hand, the dimensional considerations upon which all the phenomenological theories are based, are too ambiguous to be fruitful for any such extension. Therefore we propose a new cascade approximation, which provides a simple closure procedure in the interaction of modes, without the mathematical involvement of most analytical theories, and, at the same time, can describe in sufficient detail the dynamics of the nonlinear transfer process. The transfer function thus determined entails a viscous cutoff at arbitrarily large wave numbers of the spectrum . Let us review the basic questions to be emphasized: The question (a) of introducing the cascade approximation as a basis of closure is discussed in Sections 2 and 3, and the question (b) of formulating the transfer function is treated in Section 4. The question (c) on the structure of the eddy viscosity needs again a closure based upon the cascade concept and is treated in Sections 5 and 6. The questions (d) and (e) on the solutions of the spectral laws are discussed in Sections 7 and 8. The viscous tail of the spectrum has a cutoff which is derived in Section 8. Finally a comparison of the present cascade theory with other theories is made in Section 9.

2. CASCADE DECOMPOSITION

Consider an incompressible turbulent fluid, and assume it to be homogeneous and isotropic. The dynamical equations governing the variable velocity $\underset{\sim}{u}$ and pressure p with a constant density ρ are the equation of Navier - Stokes and the equation of continuity:

[16] Stevens A. Orszag, Some Insights into the Analytical Theories of Turbulence, in Abstract Symposium on Turbulence, Seattle, Washington, 23-27 June 1969, Boeing Scientific Research Lab. Document DI-82-853, p V1. ed. Yih-Ho Pao.

$$\frac{d\underset{\sim}{u}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u}$$

$$\nabla\cdot\underset{\sim}{u} = 0 \tag{1}$$

where ν is the kinematic viscosity, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla$$

Instead of studying the evolution of individual Fourier modes of $\underset{\sim}{u}$ and p, which would call for more information than is needed in the study of the energy spectral distribution, we bunch the modes into two groups for the sake of simplicity, and write

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{u}^{\circ} + \underset{\sim}{u}' \tag{2a}$$

with

$$\underset{\sim}{u}_0 = \int_0^k d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\underset{\sim}{x}}\ \underset{\sim}{u}(\underset{\sim}{k}) \tag{2b}$$

$$\underset{\sim}{u}' = \int_k^{\infty} d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\underset{\sim}{x}}\ \underset{\sim}{u}(\underset{\sim}{k}) \tag{2c}$$

As k is taken as an independent variable in the function $u(\underset{\sim}{k})$, it remains an independent variable in the new functions $\underset{\sim}{u}^{\circ}$ and $\underset{\sim}{u}'$ which simply represent certain sums of $\underset{\sim}{u}(\underset{\sim}{k})$. The integrations

$$\int_0^k d\underset{\sim}{k}\ \cdots \qquad \text{and} \qquad \int_k^{\infty} d\underset{\sim}{k}\ \cdots$$

denote volume integrals in the wave number space, within and outside a sphere of radius k respectively. The same notations are valid for p, It is to be noted that (2b) and (2c) can also be written as

$$\underset{\sim}{u}^{\circ}(\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\underset{\sim}{x}}\, \underset{\sim}{u}^{\circ}(\underset{\sim}{k})$$

$$\underset{\sim}{u}'(\underset{\sim}{x}) = \int_{-\infty}^{\infty} d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\underset{\sim}{x}}\, \underset{\sim}{u}'(\underset{\sim}{k})$$

where $\underset{\sim}{u}^{\circ}(\underset{\sim}{k})$ and $\underset{\sim}{u}'(\underset{\sim}{k})$ retain the values of $\underset{\sim}{u}(\underset{\sim}{k})$ but truncated in the appropriate fashion as indicated by (2b) and (2c).

As a result of such bunchings, we expect that certain intrinsic averaging or randomnization will shape up a process with a more characteristic and distinct statistical behavior, which is more suitable for physical approximations. More specifically, we expect that the equation determining $\underset{\sim}{u}^{\circ}$ will provide the evolution of the portion of the energy spectrum F(k) between 0 and k, i.e. the evolution of

$$\int_{0}^{k} dk\; F(k)$$

while the equation for $\underset{\sim}{u}'$ will determine the transport properties of the medium in which $\underset{\sim}{u}^{\circ}$ evolves.

3. BASIC ASSUMPTIONS

We list the following assumptions:

(i) The fluid is incompressible. We apply the conditions of isotropy and homogeneity for the quantities $\underset{\sim}{u}$ and $\underset{\sim}{u}^{\circ}$, since there exists no motion of larger scales which may alter these conditions. For the motion $\underset{\sim}{u}'$ of smaller scales, we assume that within the length of such small scale fluctuations, the motion is statistically homogeneous, i.e. obeying the condition of local homogeneity.

(ii) The large scale velocity $\underset{\sim}{u}^{o}$ varies very slowly compared to the rapidly varying smaller scale fluctuation $\underset{\sim}{u}'$, so that an average within a length scale comparable to $2\pi/k$, (k = a variable wave number separating the two motions), will not alter $\underset{\sim}{u}^{o}$, but will eliminate $\underset{\sim}{u}'$:

$$\langle \underset{\sim}{u}^{o} \rangle = \underset{\sim}{u}^{o}, \qquad \langle \underset{\sim}{u}' \rangle = 0 \tag{3}$$

where the angular brackets denote such an average. This is referred to as the quasi-stationary condition.

(iii) The time development of the small scale motion $\underset{\sim}{u}'$ will depend on its interaction with the larger scale motion $\underset{\sim}{u}^{o}$ under the form of a shear. Under the Boussinesq approximation, it is assumed that such an interaction plays a dominant role in the development of $\underset{\sim}{u}'$, enabling us to neglect the effect of pressure fluctuation $\nabla p'$ which merely randomnizes the energy in all directions.

(iv) It is known that a nonlinear system describing a velocity $\underset{\sim}{u}$ generates a hierarchy of equations, and that its solution necessitates an assumption of closure. Consequently, when the velocity $\underset{\sim}{u}$ is decomposed into $\underset{\sim}{u}^{o}$ and $\underset{\sim}{u}'$ as in (2a), there entails a nonlinearity in both the equation for $\underset{\sim}{u}^{o}$ and $\underset{\sim}{u}'$. The condition (i) of a homogeneous $\underset{\sim}{u}^{o}$ helps in closing the hierarchy in $\underset{\sim}{u}^{o}$; but since $\underset{\sim}{u}^{o}$ and $\underset{\sim}{u}'$ are coupled, it is necessary to closure the hierarchy generated by the nonlinearity in $\underset{\sim}{u}'$ too. The latter nonlinearity arises from the role of the streaming velocity u^* in the Lagrangian integration of a fluid element of velocity $\underset{\sim}{u}'$. If such a streaming velocity μ^* can be explicitly expressed in terms of a spectral function without its determination through a higher order equation, then the closure is achieved.

The assumptions (i) and (ii) are generally adopted in the study of an isotropic and homogeneous turbulence. The assumption (iii) is also used in the Burgers'[17] model. The assumption (iv) is new and is fundamental to securing the closures.

4. STRUCTURE OF THE TRANSFER FUNCTION, AND DERIVATION OF THE EQUATION FOR THE SPECTRAL DISTRIBUTION

By applying the average, as defined by (3), to (1) and using the notations (2), we obtain an equation for $\underset{\sim}{u}^{\circ}$ and hence $\overline{\underset{\sim}{u}^{\circ 2}}$. A subsequent subtraction of such an equation for $\underset{\sim}{u}^{\circ}$ from (1) will yield an equation for $\underset{\sim}{u}'$. Thus we write them as follows:

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\underset{\sim}{u}^{\circ 2}} = -\nu R^{\circ} - \overline{u^{\circ}_i \frac{\partial}{\partial x_j}\langle u'_i u'_j\rangle} \tag{4}$$

and

$$\frac{d\underset{\sim}{u}'}{dt} + \alpha\,\underset{\sim}{u}' = -(\underset{\sim}{u}'\cdot\nabla)\underset{\sim}{u}^{\circ} \tag{5}$$

where

$$R^{\circ} = \overline{(\partial u^{\circ}_i/\partial x_j)^2} \tag{6}$$

is a vorticity function. Similarly $R' = \langle(\partial u'_i/\partial x_j)^2\rangle$, $R = R^{\circ} + R'$.

The coefficient α will be determined approximately in the following lines.

According to the Heisenberg[1,2] hypothesis the equation for the energy dissipation (4) can be written as

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\overline{\underset{\sim}{u}^{\circ 2}} &= -(\nu+\nu_k)R^{\circ}\\ &= -\nu(R-R') - \nu_k R^{\circ}\end{aligned}$$

[17] J. M. Burgers, in Advances in Applied Mechanics, R. Von Mises and Th. Von Kármán. Eds. (Academic Press Inc., New York, 1948) Vol. 1, p. 171.

or

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\overline{\underset{\sim}{u}^2} - \langle \underset{\sim}{u}'^2 \rangle\right) = -\nu R + \nu R' - \nu_k R^o$$

It follows

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \underset{\sim}{u}'^2 \rangle = -\nu R' + \nu_k R^o$$
$$= -(\nu - \gamma \nu_k) R', \qquad \gamma = R^o/R'$$

We see that the dissipation of $\frac{1}{2}\langle \underset{\sim}{u}'^2 \rangle$ occurs at a lesser rate than $\nu R'$, i.e. at a rate $\beta^{-1}\nu R'$ with $\beta > 1$. We then have

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \underset{\sim}{u}'^2 \rangle = -\beta^{-1}\nu R'$$
$$= -\alpha \langle \underset{\sim}{u}'^2 \rangle$$

where

$$\alpha = \nu R' / \beta \langle \underset{\sim}{u}'^2 \rangle \tag{7a}$$

is the damping coefficient in (5), and

$$\beta^{-1} = 1 - \nu_k R^o / \nu R' \tag{7b}$$

approximated as a constant, will be estimated in Section 8.

In (4), we have introducbd the bar as an average taken over a length tending to infinity, as differentiated from the average (3) taken over a length scale $2\pi/k$. We have neglected the terms

$$\frac{1}{2}\nabla\cdot\overline{\underset{\sim}{u}^o\,\underset{\sim}{u}^{o2}}, \qquad \nabla\cdot\overline{p\underset{\sim}{u}^o}, \qquad \nu\nabla^2\overline{\underset{\sim}{u}^{o2}}$$

As a result of the assumption (i) on homogeneity.

In arriving at (5), we have neglected the term $\langle (\underset{\sim}{u}' \cdot \nabla) \underset{\sim}{u}' \rangle$ under the quasi-stationary assumption (ii), but retained the nonlinear term $(\underset{\sim}{u}' \cdot \nabla) \underset{\sim}{u}'$ embedded in d/dt. In addition, we have neglected the pressure effect under the assumption (iii) of Boussinesq approximation, and adopted the approximate damping α in (5) and (7a).

The completed system of equations (4) and (5) are nonlinear. It is seen that the development of the energy $\frac{1}{2} \overline{\underset{\sim}{u}^{o2}}$ is governed by a viscous dissipation νR^o in (4), and by an eddy dissipation caused by the smaller scale fluctuations playing the role of a shear stress $\langle u_i' u_j' \rangle$.
The motions $\underset{\sim}{u}^o$ and $\underset{\sim}{u}'$ are coupled through (5), in agreement with the concept of mixing by turbulent shear, initiated by Boussinesq.

As a solution of (5), we find

$$u_i'(t) = - \frac{\partial u_i^o}{\partial x_s} \int_0^t dt' \, e^{-\alpha(t-t')} u_s'(t') \tag{8}$$

assuming $\partial u_i^o / \partial x_j$ to be quasi-stationary, according to assumption (ii). In (8) we have written $\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{u}(t-t')$, and the integration is made along the trajectory of the small scale element $\underset{\sim}{u}'$, which is transported by $\underset{\sim}{u}$. From (8) it follows

$$\langle u_i' u_j' \rangle = - \frac{\partial u_i^o}{\partial x_s} \int_0^t d\tau \, e^{-\alpha \tau} \langle u_s'(t-\tau) u_j'(t) \rangle$$

Since the duration of the correlation is small, we can replace the upper limit of integration by ∞, without altering the values of the integrals. Further the integral, representing an eddy viscosity ν_k caused by small scale fluctuations, can be assumed isotropic and locally homogeneous, see assumption (i); therefore we can write for large t:

$$\langle u_i' \, u_j' \rangle = -\frac{1}{3} \frac{\partial u_i^o}{\partial x_s} \int_0^\infty d\tau \, e^{-\alpha\tau} \langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) \rangle \, \delta_{sj}$$

$$= \nu_k \frac{\partial u_i^o}{\partial x_j} \tag{9a}$$

where

$$\nu_k = \frac{1}{6} \int_{-\infty}^{\infty} d\tau \, e^{-\alpha\tau} \langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) \rangle \tag{9b}$$

is an eddy viscosity from small scale fluctuations. Equation (9a) is a relation between the fluctuation and the gradient of the background larger scale flow, through the intermediary of an eddy viscosity. This result is in agreement with the Boussinesq's mixing length theory.

Finally upon substituting (9a) into (4), we reduce (4) to

$$\frac{1}{2} \frac{\partial}{\partial t} \overline{\underset{\sim}{u}^{o2}} = -\nu R^o + \nu_k \overline{u_i^o \frac{\partial^2 u_i^o}{\partial x_j^2}}$$

$$= -\nu R^o - \nu_k \left[\overline{\left(\frac{\partial u_i^o}{\partial x_j} \right)^2} - \frac{1}{2} \nabla^2 \overline{(u^{o2})} \right]$$

$$= -(\nu + \nu_k) R^o \tag{10}$$

with the aid of assumption of homogeneity (i).

We consider now the equilibrium range of the spectrum F(k). In this range valid for large wave numbers, the term

$$\frac{\partial}{\partial t}\int_k^{\infty} dk\, F$$

is very small, therefore,

$$\frac{\partial}{\partial t}\int_0^k dk\, F \cong \frac{1}{2}\frac{\partial}{\partial t}\overline{u^{D2}}$$
$$= \varepsilon \equiv \nu R$$

reducing (10) to

$$(\nu + \nu_k)R^{D} = \varepsilon \tag{11}$$

an equation similar to the equation (11) proposed by Heisenberg[1,2].

5. RELATION BETWEEN THE SPECTRUM AND THE FOURIER COMPONENTS OF VELOCITIES

We write the Fourier transform in the space $d\Omega \equiv d\omega\, dk$

$$u(t,x) = \int_{-\infty}^{\infty} d\Omega\, e^{-i(\omega t - k x)}\, u(\omega, k)$$
$$u(t',x') = \int_{-\infty}^{\infty} d\Omega'\, e^{-i(\omega' t - k' x) + i(\omega' \tau - k' r)}\, u(\omega', k')$$

where $t' = t - \tau$, $x' = x - r$. Let us form the time and space integral of $u(t',x')\cdot u(t,x)$:

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx\;\; u(t',x')\cdot u(t,x)$$

$$\int_{-\infty}^{\infty} d\underset{\sim}{\Omega}' \int_{-\infty}^{\infty} d\underset{\sim}{\Omega}\; e^{i(\omega'\tau - \underset{\sim}{k}'\underset{\sim}{r})}\, \underset{\sim}{u}(\omega', \underset{\sim}{k}')\cdot \underset{\sim}{u}(\omega, \underset{\sim}{k})$$
$$\times \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\underset{\sim}{x}\; e^{-i(\omega+\omega')t + i(\underset{\sim}{k}+\underset{\sim}{k}')\underset{\sim}{x}} \tag{12}$$

Noting that

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega+\omega')t} = 2\pi\, \delta(\omega+\omega')$$

and similarly for the space integral, we transform (12) to

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\underset{\sim}{x}\;\; \underset{\sim}{u}(t', \underset{\sim}{x}')\cdot \underset{\sim}{u}(t, \underset{\sim}{x})$$
$$= (2\pi)^4 \int_{-\infty}^{\infty} d\underset{\sim}{\Omega}\;\; |\underset{\sim}{u}(\omega, \underset{\sim}{k})|^2 e^{-i(\omega\tau - \underset{\sim}{k}\underset{\sim}{r})} \tag{13}$$

degenerating to the Parseval theorem when $\tau = 0$,

For the sake of convenience of Fourier transform, the function $\underset{\sim}{u}(t, \underset{\sim}{x})$ is supposed to be bounded within a time integral $2T$ and a length integral $2X$, called intervals of truncation, with

$$\chi = \frac{\pi}{T}\left(\frac{\pi}{X}\right)^3$$

then (13) can be rewritten as

$$\overline{\underset{\sim}{u}(t', \underset{\sim}{x}')\cdot \underset{\sim}{u}(t, \underset{\sim}{x})} = \chi \int_{-\infty}^{\infty} d\underset{\sim}{\Omega}\;\; |\underset{\sim}{u}(\omega, \underset{\sim}{k})|^2 e^{-i(\omega\tau - \underset{\sim}{k}\underset{\sim}{r})} \tag{14}$$

where the bar represents a time average and a space average within the above intervals of truncation. With the special case of $\tau = 0$, we reduce to

$$\overline{\underset{\sim}{u}^2} = \chi \int_{-\infty}^{\infty} d\underset{\sim}{k} \int_{-\infty}^{\infty} d\omega \; \left|\underset{\sim}{u}(\omega, \underset{\sim}{k})\right|^2$$

$$= \chi \int_0^{\infty} dk \; 4\pi k^2 \int_{-\infty}^{\infty} d\omega \; \left|\underset{\sim}{u}(\omega, \underset{\sim}{k})\right|^2$$

$$= 2 \int_0^{\infty} dk \; F(k)$$

where

$$F(k) = \int_{-\infty}^{\infty} d\omega \; 2\pi k^2 \chi \left|\underset{\sim}{u}(\omega, \underset{\sim}{k})\right|^2 \tag{15}$$

6. EDDY VISCOSITY

a) Effect of Viscous Damping in the Eddy Viscosity

In all the theories reviewed in Section 1, the effect of viscous damping does not enter in the eddy viscosity. In the present Section, we shall include such an effect in the derivation of the eddy viscosity, as it is important at large wave numbers and is expected to play an essential role in the convergence of the spectrum.

With the aid of (14), and on account of the even value of the integrand, we can rewrite (9b) as

$$\nu_k = \frac{1}{6} \int_{-\infty}^{\infty} d\tau \; e^{-\alpha|\tau|} \langle \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) \rangle$$

$$= \frac{\chi}{6} \int_{-\infty}^{\infty} d\underset{\sim}{\Omega} \; \left|\underset{\sim}{u}'(\omega, \underset{\sim}{k})\right|^2 \int_{-\infty}^{\infty} d\tau \; e^{-i(\omega - \underset{\sim}{k}\underset{\sim}{u})\tau - \alpha|\tau|}$$

$$= \frac{\chi}{6} \int_{-\infty}^{\infty} d\omega \int_k^{\infty} dk \; 2\pi k^2 \; \left|\underset{\sim}{u}(\omega, \underset{\sim}{k})\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-1}^{+1} d\mu \; e^{-i(\omega - k u \mu)\tau - \alpha|\tau|} \tag{16}$$

where $\mu = \cos\theta$ in the spherical polar coordinates k, θ, ϕ .

We shall first perform the integration with respect to μ, giving

$$\int_{-\infty}^{\infty} d\tau \int_{-1}^{+1} d\mu \, e^{-i(\omega - ku\mu) - \alpha|\tau|}$$

$$= 2 \int_{-\infty}^{\infty} d\tau \, \frac{\sin ku\tau}{ku\tau} \, e^{-i\omega\tau - \alpha|\tau|}$$

The Lagrangian time integration is very complicate, since μ depends on τ too. Consider first

$$\varphi(\omega, \omega^*/\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \, \frac{\sin\omega^*\tau}{\tau} e^{-i\omega\tau - \alpha|\tau|}$$

with

$$\omega^* \equiv ku^*$$

where u^* is a streaming velocity independent of τ. We have

$$\varphi(\omega, \omega^*/\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \, e^{-\alpha|\tau|} \frac{1}{2} \left[\frac{\sin(\omega^*+\omega)\tau}{\tau} + \frac{\sin(\omega^*-\omega)\tau}{\tau} \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \, e^{-\alpha|\tau|} \int_{\omega-\omega^*}^{\omega+\omega^*} d\omega' \cos\omega'\tau$$

$$= \frac{1}{\pi} \left[\arctan \frac{\omega^*+\omega}{\alpha} + \arctan \frac{\omega^*-\omega}{\alpha} \right] \tag{17a}$$

Upon inspecting (17a), we notice that the damping factor $\varphi(\omega, \omega^*/\alpha)$ becomes a unit function for $\alpha = 0$:

$$\lim_{\alpha=0} \varphi(\omega, \omega^*/\alpha) = \begin{cases} 1, & \text{for} \quad \omega < \omega^* \\ 0, & \text{for} \quad \omega > \omega^* \end{cases}$$

thus it selects the contributions from the low frequencies $\omega<\omega^*$, and decreases monotonously with increasing α . Hence (17a) and (18a) simplify to

$$\varphi(\omega^*/\alpha) \equiv \varphi(\omega<\omega^*, \omega^*/\alpha)$$

$$= \frac{2}{\pi} \arctan(\omega^*/\alpha) \qquad (17b)$$

It is to be remarked that the multiples of $\pi/2$ in the $\arctan(\omega^*/\alpha)$ are not taken in order to conform the value unity of (17b) to its equivalent

$$\lim_{\alpha=0} \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \, e^{-\alpha|\tau|} \frac{\sin \omega^* \tau}{\tau} = 1$$

when $\alpha = 0$.

In view of the property of selecting low frequencies, in the Lagrangian time integration as mentioned above, we can simplify (16) to

$$\nu_k = \frac{\pi \chi}{3} \int_k^{\infty} dk \; 4\pi k^2 \int_0^{\infty} d\omega \; |\underset{\sim}{u}(\omega, \underset{\sim}{k})|^2$$
$$\times \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \; \frac{\sin k u \tau}{k u \tau} e^{-i\omega\tau - \alpha|\tau|}$$

$$\cong \frac{\pi \chi}{3} \int_k^{\infty} dk \left[4\pi k^2 \int_0^{\infty} d\omega \, \left| \underset{\sim}{u}(\omega, \underset{\sim}{k}) \right|^2 \omega^{*-1} \varphi(\omega, \omega^*/\alpha) \right]$$

where ku is replaced by ω^* varying slowly in time. It is to be noted that ω^* still depends on k and F.

Since φ is truncated, we can further write

$$\nu_k = \frac{\pi \chi}{3} \int_k^{\infty} dk \left[\frac{4\pi k^2}{\omega^*} \varphi(\omega^*/\alpha) \int_0^{\omega^*} d\omega \, \left| \underset{\sim}{u}(\omega, \underset{\sim}{k}) \right|^2 \right]$$

Although the relation between ω and k forms a difficult nonlinear dispersion relation, worthy of a separate study, it is safe to assume that the frequency is contributed by the convection of a scale k by a streaming velocity u*, so that the integral

$$\int_0^{\omega^*} d\omega \cdots$$

may encompass the major frequency contributions. Thus without much error, we can replace the upper integration limit by ∞, giving

$$\nu_k = \frac{\pi \chi}{3} \int_k^{\infty} dk \left[\frac{2\pi k^2}{\omega^*} \varphi(\omega^*/\alpha) \int_{-\infty}^{\infty} d\omega \, \left| \underset{\sim}{u}(\omega, \underset{\sim}{k}) \right|^2 \right]$$

$$= \frac{\pi}{3} \int_k^{\infty} dk \;\; F(k) \, \omega^{*-1} \, \varphi(\omega^*/\alpha) \qquad (18c)$$

with the use of representation (15). It is to be recalled that the effect of the viscous damping resides in φ, as given by (17b).

b) Relaxation Frequency ω^*

The relaxation frequency determines the time necessary for the energy Fdk between k and k + dk to relax and dissipate through the corresponding vorticity $2k^2$ Fdk. Thus we can write, in analogy to (10),

$$\omega^* F = (\nu + \nu_k)\, 2k^2 F \tag{20}$$

By neglecting the effect of the molecular viscosity and with the use of (18c), we can transform (20) into

$$\frac{\omega^*}{2k^2} = \frac{\pi}{3} \int_k^\infty dk \;\; F/\omega^*$$

yielding the solution

$$\omega^* = \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} k^2 \left[\int_k^\infty dk \;\; F/k^2\right] \tag{21}$$

and

$$u^* = \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} k \left[\int_k^\infty dk \;\; F/k^2\right]$$

Thus the streaming velocity responsible for the convection of the eddies of wave number k in a strong interaction depends on the local behavior characterized by F and k. The dynamics represented by (20), which describes such an interaction, is again based on a cascade concept, similar to that used in (10).

7. KOLMOGOROFF SOLUTION OF THE SPECTRAL DISTRIBUTION

In the inertial subrange we can neglect $\nu \ll \nu_k$ in (11), and approximate the damping factor φ to unity in (18c), so that (11) simplifies to the integral equation

$$\frac{\pi}{3} R^{\circ} \int_{k}^{\infty} dk\ F/\omega^{*} = \varepsilon \tag{22}$$

with ω^{*} given by (21), yielding the solution

$$F = A\,\varepsilon^{2/3} k^{-5/3} \tag{23a}$$

with

$$A = \left(32/3\sqrt{\pi}\right)^{2/3} \tag{23b}$$

Although the integral equation (22) differs from that proposed gy Heisenberg[1,2], the power law $k^{-5/3}$ obtained in (23a) is in agreement with the Kolmogoroff[7] and Heisenberg[1,2] theories. The present theory derives the transfer function without the empirical and dimensional arguments, and determines the numerical coefficient (23b).

8. TAIL OF THE VISCOUS SUBRANGE

In order to obtain a solution of (11) in the viscous subrange, we differentiate (11) with respect to k giving

$$R^{\circ} \frac{d\nu_k}{dk} + (\nu + \nu_k) \frac{dR^{\circ}}{dk} = 0$$

Since the viscous subrange occurs at large k, we can replace R_o by $R = R_o(k \to \infty)$, and neglect $\nu_k \ll \nu$ reducing the differential equation to

$$R \frac{d\nu_k}{dk} + \nu \frac{dR^0}{dk} = 0$$

or, in terms of F, with the aid of (17b), (18c) and (21),

$$k^8 \int_k^\infty dk \, (F/k^2) = \frac{\pi}{3} \left(R/4\nu \right)^2 \varphi^2 \tag{24}$$

where φ is a cutoff function given by (17b).

Assume a solution

$$F = \frac{\pi}{6} (R/\nu)^2 k^{-7} \psi(k/k_\nu) \tag{25}$$

with

$$k_\nu = (R/\nu^3)^{1/4}$$

The solution

$$\frac{\pi}{6} (R/\nu)^2 k^{-7}$$

satisfies the equation (24) when φ is unity, and ψ is a dimensionaless function of k/k_ν, to be determined by the integral equation (24), which reduces to

$$\psi = \varphi^2$$

Now by using (7), (21) and (24), we find

$$\omega^*/\alpha = \frac{\pi}{3}\beta\, F k^{-1} k_\nu^4 \varphi R'^{-1}$$

which reduces (17b) to

$$\varphi = \frac{2}{\pi} \arctan\left(\frac{\pi}{3}\beta F k^{-1} k_\nu^4 \varphi R'^{-1}\right)$$

or inversely

$$R' = \frac{2}{3}\beta\, F k^{-1} k_\nu^4 \quad \frac{\pi \varphi/2}{\tan(\pi\varphi/2)}$$

Upon differentiating with k, we have

$$-2k^2 F = \frac{2}{3}\beta F k^{-1} k_\nu^4 \varphi^{-1} \frac{d\varphi}{dk}$$

$$-\left\{\frac{2}{3}\beta F k^{-1} k_\nu^4 \varphi^{-1}\frac{d\varphi}{dk}\left[1 - \frac{\pi\varphi/2}{\tan(\pi\varphi/2)} + \frac{(\pi\varphi/2)^2}{1+\tan^2(\pi\varphi/2)}\right]\right.$$

$$\left. - \frac{2}{3}\beta \frac{d}{dk}(F k^{-1}) k_\nu^4 \frac{\pi\varphi/2}{\tan(\pi\varphi/2)}\right\}$$

For $k/k_\nu > 1$, the terms between the brackets are negligible, simplifying to

$$\frac{1}{\varphi}\frac{d\varphi}{dk} = -3\beta^{-1} k^3 k_\nu^{-4}$$

yielding the solution

$$\psi = \varphi^2 = \exp\left[-\frac{3}{2\beta}(k/k_\nu)^4\right]$$

Hence (25) becomes

$$F = \frac{1}{2}\left(\frac{R}{\nu}\right)^2 k^{-7} \exp\left[-\frac{3}{2\beta}(k/k_\nu)^4\right] \tag{27}$$

If $0.5k_\nu$ is the wave number of transition from the inertial to the viscous subranges[18], the exponential cutoff is effective at a wave number larger than k_ν, since $\beta > 1$. Hence we conclude that the viscosity effect in the expression (18c) for the eddy viscosity will provide a cutoff of the spectrum at large k.

The exponential tail of (27) secures the convergence of any high order velocity derivative. Such a convergence was absent in the power law k^{-7} of the Heisenberg[1,2] solution.

For small k, the value of β is nearly unity. We shall make an estimate of the value of β near $k = k_\nu \to \infty$. We write in terms of F

$$\nu_k \frac{R^\circ}{R'} = \frac{\int_0^k dk\, k^2 F}{\int_k^\infty dk\, k^2 F} \frac{\pi}{3}\int_k^\infty dk \frac{F}{\omega^*}\varphi$$

$$= \frac{\pi}{6} R^\circ \varphi \left(k^2 \omega^*\right)^{-1}, \quad \text{at } k \cong k_\nu$$

and according to (21) and (25)

$$\frac{\nu_k}{\nu}\frac{R^\circ}{R'} \cong \frac{R^\circ(k = k_\nu)}{R}$$

[18] J. O. Hinze, Turbulence (McGraw-Hill Book Company, Inc., New York, 1959), Chap. 3, p. 195, formula (3-128).

Hence for $k \to k_\nu \to \infty$

$$\beta = R/R'(k = k_\nu) \doteqdot 4$$

Thus the cutoff occurs at $k_{cut} \doteq (8/3)^{1/4} k_\nu$.

9. COMPARISON WITH OTHER THEORIES

a. The present cascade model yields an equation (10) for the development of the spectral distribution, the equation is similar to that proposed by Heisenberg[1,2]. However, the eddy viscosity is derived in (18c) and (21), and takes a form at variance with the one proposed by Heisenberg[1,2] on a dimensional argument. The equation (10) yields a solution (23a) for the inertial subrange, and a solution (27) for the viscous subrange. The power law in (23a) is in agreement with the Kolmogoroff[7] - Heisenberg[1,2] spectrum. The numerical constant (23b) is derived. The viscous law (27) has the appearance of Heisenberg's[1,2] law, but we find an exponential cutoff k^{-7} at $k = 1.3\ k_\nu$. The cutoff does not follow immediately the $k^{-5/3}$ law, as suggested by several authors[4,5]. If the viscous drop occurs at $k_d = 0.5\ k_\nu$, there must exist a narrow range

$$k_d < k < k_{cut}$$

i.e.

$$0.5 < k/k_\nu < 1.3$$

within which Heisenberg's[1,2] k^{-7} law should hold.

The formula (27) gives also the numerical coefficient which was left undetermined in the Heisenberg[1,2] theory.

b. The eddy viscosity (18c) depends on the relaxation frequency ω^*. The latter has the same dimension as $R^{\circ 1/2}$. If we approximate (18c) by

$$\nu_k = \frac{\pi}{3} R^{\circ -\frac{1}{2}} \int_k^\infty dk\, F$$

without altering the dimensional structure, then the equation (11) for the spectral equation reduces to

$$\nu R^\circ + \frac{\pi}{3} R^{\circ \frac{1}{2}} \int_k^\infty dk\, F = \varepsilon$$

which is equivalent to the Obukhov[3] equation, yielding also the Kolmogoroff[7] law in the inertial subrange.

c. The relaxation frequency $\omega^* \equiv ku^*$ has been determined consistently by the cascade concept described by (20); consequently it is due to the convection of a small scale motion by a streaming velocity u^* set up by a local cascade, in the sense that a strong interaction should operate between the two scales. Instead of such a strong interaction, we may assume a weak interaction, so that the streaming velocity u^* does not depend on the local cascade, but is a measure of the strength of turbulence at large. Under such a circumstance, (18c) approximates to

$$\nu_k = \frac{\pi}{3} u^{*-1} \int_k^\infty dk\, F k^{-1}, \qquad \varphi = 1$$

reducing (10) to:

$$\left(\nu + \frac{\pi}{3u^*}\int_k^{\infty} dk \; F k^{-1}\right) R^0 = \varepsilon$$

Its solution for the inertial subrange is

$$F = A\,(\varepsilon\, u^*)^{\frac{1}{2}}\; k^{-3/2}, \qquad A = \pi^{-\frac{1}{2}}(3/2)^{3/2}$$

and is found to be equivalent to the result of Kraichnan[19], using the approximation of stochastic direct interaction.

The method does not give a solution for the viscous subrange.

ACKNOWLEDGMENT

This research was supported by the National Aeronautics and Space Administration, under Contract # NGR 33-013-034.

[19] R. Kraichnan, J. Fluid Mech. 5, 497 (1959).